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LETTER TO THE EDITOR

Dirac quantisation of massive spin-$\frac{3}{2}$ field

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Abstract. We apply Dirac's Hamiltonian method to the classical Lagrangian of a free massive anticommuting spin-$\frac{3}{2}$ field. The anticommutation rules needed in the transition to the quantum theory are obtained.

The classically anticommuting spin-$\frac{3}{2}$ field plays a significant role in the formulation of the locally supersymmetric theory of supergravity (Freedman *et al* 1976, Deser and Zumino 1976). Incidently, Dirac's Hamiltonian method of quantising constrained systems (Dirac 1964) has been applied to the anticommuting massless spin-$\frac{3}{2}$ field both free (Senjanovic 1977) and interacting with gravity (Teitelboim 1977, Fradkin and Vasiliev 1977). However, we have not seen this method applied to the massive field, although the Hamiltonian formulation of massive supergravity (Freedman and Das 1977, Townsend 1977, Baaklini 1977b) has been approached (Deser *et al* 1977). Moreover, the anticommutation rules needed in the transition to the quantum theory have not been given.

Our purpose in this Letter is to apply Dirac's method to the massive anticommuting spin-$\frac{3}{2}$ field and obtain the anticommutation rules. In the following, we start from the classical Lagrangian of the massive free anticommuting spin-$\frac{3}{2}$ field. We go over to the Hamiltonian formulation. The second class constraints obtained in the procedure are used to define, through the Dirac brackets (Dirac 1964), the anticommutation rules of the quantum theory.

The classical Lagrangian of an anticommuting, free and massive spin-$\frac{3}{2}$ field, in the form used in massive supergravity (Freedman and Das 1977, Townsend 1977, Baaklini 1977b) is given by

$$L=\int \mathrm{d}^3x\,(\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\bar{\psi}_\mu \mathrm{i}\gamma_\nu\gamma_5\partial_\lambda\psi_\rho+\tfrac{1}{2}m\bar{\psi}_\mu \mathrm{i}\sigma^{\mu\nu}\psi_\nu). \tag{1}$$

Our conventions are

$$\begin{aligned} &\eta_{\mu\nu}=\mathrm{diag}(+\,-\,-\,-), && \epsilon^{0123}=1 \\ &\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3, && \sigma_{\mu\nu}=\tfrac{1}{2}\mathrm{i}[\gamma_\mu,\gamma_\nu] \end{aligned} \tag{2}$$

and $\psi^\alpha_\mu(x)$ is a classically anticommuting Majorana field obeying the anticommutation and reality conditions

$$[\psi^\alpha_\mu(x),\psi^\beta_\nu(x)]_+=0, \qquad \psi_\mu=C\bar{\psi}_\mu \tag{3}$$

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where C is the charge conjugation matrix. Note that the case of a complex Dirac field can easily be done, taking the latter as a complex combination of two Majorana fields.

The canonical momenta $\bar{\eta}^{\mu}_{\alpha}(x)$ conjugate to $\psi^{\alpha}_{\mu}(x)$ are defined by making the variation

$$\delta L = \int \mathrm{d}^3x\, \bar{\eta}^{\mu}(x)\, \delta\dot{\psi}_{\mu}(x) \tag{4}$$

where the dot means differentiation with respect to time.

Splitting the Lagrangian (1) into space and time, we obtain

$$L = \int \mathrm{d}^3x\, (\tfrac{1}{2}\mathrm{i}\epsilon_{ijk}\bar{\psi}_i\gamma_j\gamma_5\dot{\psi}_k - \tfrac{1}{2}\mathrm{i}\epsilon_{ijk}\bar{\psi}_i\gamma_0\gamma_5\partial_j\psi_k + \epsilon_{ijk}\bar{\psi}_0\mathrm{i}\gamma_i\gamma_5\partial_j\psi_k - m\bar{\psi}_0\mathrm{i}\sigma_{0i}\psi_i + \tfrac{1}{2}m\bar{\psi}_i\mathrm{i}\sigma_{ij}\psi_j). \tag{5}$$

Hence we have for the conjugate momenta (4),

$$\bar{\eta}^{0}_{\alpha}(x) = 0 \qquad \bar{\eta}^{k}_{\alpha}(x) = \tfrac{1}{2}\epsilon^{kij}(\bar{\psi}_i(x)\mathrm{i}\gamma_j\gamma_5)_{\alpha}. \tag{6}$$

The classically anticommuting variables $\psi^{\alpha}_{\mu}(x)$ and their conjugate momenta $\bar{\eta}^{\mu}_{\alpha}(x)$ are regarded as odd elements of a Grassman algebra (Berezin 1966). However, one can still define consistent classical Poisson brackets for them (antibrackets) (Casalbuoni 1976, Baaklini 1977a).

We define the fundamental Poisson antibrackets

$$\{\bar{\eta}^{\mu}_{\alpha}(x), \bar{\psi}_{\nu\beta}(y)\} = \delta^{\mu}_{\nu}C^{-1}_{\alpha\beta}\,\delta^3(x-y). \tag{7}$$

The Hamiltonian is

$$\begin{aligned} H &= \int \mathrm{d}^3x\, \bar{\eta}^{\mu}\dot{\psi}_{\mu} - L \\ &= \int \mathrm{d}^3x\, [\tfrac{1}{2}\epsilon_{ijk}\bar{\psi}_i\mathrm{i}\gamma_0\gamma_5\partial_j\psi_k - \tfrac{1}{2}m\bar{\psi}_i\mathrm{i}\sigma_{ij}\psi_j - \mathrm{i}\bar{\psi}_0(\epsilon_{ijk}\gamma_i\gamma_5\partial_j\psi_k - m\sigma_{0i}\psi_i)]. \end{aligned} \tag{8}$$

Now from equation (6) we obtain the weakly vanishing (≈ 0) primary constraints

$$\begin{aligned} \bar{K}^{0}_{\alpha} &\equiv \bar{\eta}^{0}_{\alpha} \approx 0 \\ \bar{K}^{k}_{\alpha} &\equiv \bar{\eta}^{k}_{\alpha} - \tfrac{1}{2}\epsilon^{kij}(\bar{\psi}_i\mathrm{i}\gamma_j\gamma_5)_{\alpha} \approx 0. \end{aligned} \tag{9}$$

These constraints are second class. The Hamiltonian (8) is defined up to the constraints (9) and hence they could be added to it via Lagrange multipliers. The latter are determined by the requirement that the constraints have vanishing Poisson brackets with the Hamiltonian (constant in time). However, this procedure involves technical difficulties and much labour. A simpler procedure is to define immediately the Dirac brackets (Dirac 1964),

$$\{f, g\}^* = \{f, g\} - \{f, K_i\}(M^{-1})_{ij}\{K_j, g\} \tag{10}$$

for all dynamical variables f and g. Here M^{-1}_{ij} is the inverse of the matrix

$$M_{ij} = \{K_i, K_j\} \tag{11}$$

for the second class constraints K_i.

After defining the new brackets (10), the second class constraints can be put strongly equal to zero. Note that this procedure can be done by iteration, taking successive subsets of the constraints.

Calculating the Poisson bracket of the constraints $\bar{K}^i_\alpha$, we obtain

$$\{\bar{K}^i_\alpha(x), \bar{K}^j_\beta(y)\} = \epsilon^{ijk}(C^{-1}\mathrm{i}\gamma_k\gamma_5)_{\alpha\beta}\,\delta^3(x-y) \equiv M^{ij}_{\alpha\beta}(x-y). \tag{12}$$

The inverse of the resulting matrix is

$$(M^{-1})^{jk}_{\beta\gamma}(x-y) = -\tfrac{1}{2}[\delta^{jk}(\mathrm{i}\gamma_0 C)_{\beta\gamma} + (\gamma_0\sigma^{jk}C)_{\beta\gamma}]\,\delta^3(x-y). \tag{13}$$

Hence, using (10), we obtain the Dirac brackets

$$\{\bar{\psi}^\alpha_i(x), \bar{\psi}^\beta_j(y)\}^* = -\tfrac{1}{2}(C^{-1}\mathrm{i}\gamma_0\gamma_j\gamma_i)^{\alpha\beta}\,\delta^3(x-y). \tag{14}$$

Thus we can eliminate $\bar{\eta}^i_\alpha(x)$ from the theory by setting $\bar{K}^i_\alpha$ equal to zero.

Requiring that the Poisson bracket of $\bar{K}^0_\alpha(x)$ with the Hamiltonian should vanish, we obtain a new constraint

$$\chi_\alpha(x) \equiv \epsilon^{ijk}(\gamma_i\gamma_5\partial_j\psi_k)_\alpha - m(\sigma_{0i}\psi_i)_\alpha \approx 0. \tag{15}$$

Observe that $\psi_0(x)$ has manifested itself as a Lagrange multiplier as noted in Deser *et al* (1977).

The secondary constraint $\chi_\alpha(x)$ is second class and can be put equal to zero in the same manner as $\bar{K}^i_\alpha(x)$. Using the Dirac bracket (14), we obtain

$$\{\chi_\alpha(x), \chi_\beta(y)\}^* = -\tfrac{3}{2}m^2(\mathrm{i}\gamma_0 C)_{\alpha\beta}\,\delta^3(x-y) \equiv N_{\alpha\beta}(x-y). \tag{16}$$

Note that in the massless case ($m=0$), $\chi_\alpha(x)$ is a first class constraint and a gauge-fixing condition would be needed (Senjanovic 1977).

The inverse of the resulting matrix (16) is

$$N^{-1}_{\beta\gamma}(y-z) = \frac{2}{3m^2}(\mathrm{i}C^{-1}\gamma_0)_{\beta\gamma}\,\delta^3(y-z). \tag{17}$$

Hence, we obtain the doubly new Dirac bracket

$$\begin{aligned}
&\{\bar{\psi}^\alpha_i(x), \bar{\psi}^\beta_j(y)\}^{**}\\
&\quad= \{\bar{\psi}^\alpha_i(x), \bar{\psi}^\beta_j(y)\}^* - \int \mathrm{d}^3w\,\mathrm{d}^3z\{\bar{\psi}^\alpha_i(x), \chi^\gamma(z)\}^* N^{-1}_{\gamma\delta}(z-w)\{\chi^\delta(w), \bar{\psi}^\beta_j(y)\}^*\\
&\quad= \Big(-\tfrac{1}{2}(C^{-1}\mathrm{i}\gamma_0\gamma_j\gamma_i)^{\alpha\beta} - \tfrac{1}{6}(\mathrm{i}C^{-1}\gamma_0\gamma_i\gamma_j)^{\alpha\beta} + \frac{2}{3m^2}(\mathrm{i}C^{-1}\gamma_0)^{\alpha\beta}\partial_i\partial_j\\
&\qquad - \frac{1}{3m}(C^{-1}\gamma_0\gamma_i)^{\alpha\beta}\partial_j - \frac{1}{3m}(C^{-1}\gamma_0\gamma_j)^{\alpha\beta}\partial_i\Big)\,\delta^3(x-y).
\end{aligned} \tag{18}$$

After obtaining equation (18), the constraint $\chi_\alpha(x)$ can be set equal to zero. The term involving $\psi_0(x)$ in the Hamiltonian (8) drops out. Hence the dynamical variables of the theory are $\psi^\alpha_i(x)$ and they obey the fundamental brackets (18).

The transition to the quantum theory (Dirac 1964) is made by regarding $\psi^\alpha_i(x)$ as operators on a state functional. They obey anticommutation rules which are equal to $(-i)$ times the brackets (18).

The path integral corresponding to the quantum theory is the conventional one. The situation is very similar to the massive vector meson theory as can be seen by comparing with the work of Senjanovic (1976).

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